SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2007
YEAR 12 Mathematics Extension 1
HSC Task #3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 70

- Attempt questions 1-3
- Start each new section of a separate writing booklet

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

QUESTION 1 (23 marks) — Marks

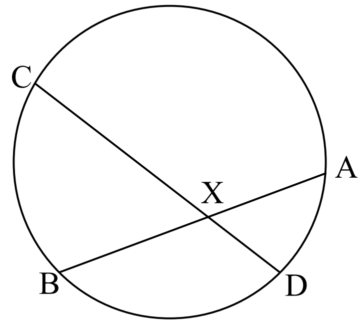
(a) Evaluate $\cos^{-1}\left(\frac{2}{5}\right)$ in radians to 4 significant figures. **1**

(b) Use the table of integrals to find

(i) $\int \frac{dx}{9+x^2}$ **1**

(ii) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{7}\right)$ **1**

(c)



If AX = 3, BX = 4 and CX = 6 find DX. **1**

(d) Find the inverse functions of the following

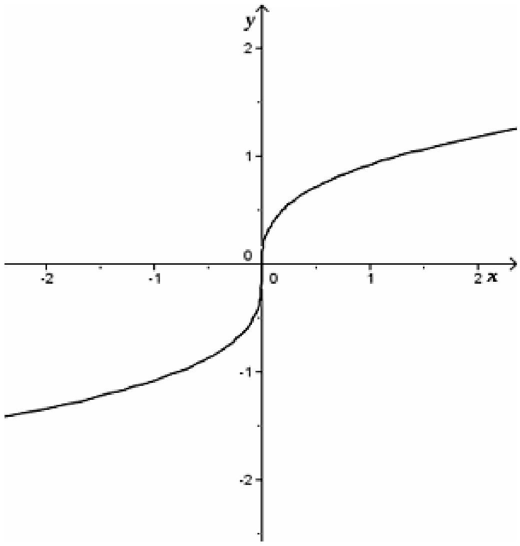
(i) $f(x) = \frac{x+1}{4}$ **1**

(ii) $g(x) = \frac{x+2}{1-x}$ **1**

(e) Evaluate $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$. **1**

(f) It is obvious that, $\sqrt{4} < \sqrt{7} < \sqrt{9}$. Use two application of the 'halving the interval' method on the equation $x^2 - 7 = 0$ to find a smaller interval containing $\sqrt{7}$. **2**

(g)



The diagram shows the graph of $y = \sqrt[3]{x}$. Copy the diagram into your examination booklet. On your diagram sketch the graph of the inverse function of $y = \sqrt[3]{x}$. **2**

(h) Find $\int 8x(x^2 + 2)^3 \, dx$ using the substitution $u = x^2 + 2$. **2**

(i) A rock is thrown into a still pond and causes a circular ripple. If the radius of the ripple in increasing a 0.5 m/s.

(i) Find the rate of change of the area as a function of the radius. **2**

(ii) How fast is the area increasing when the radius is 10 m? **1**

(j)

(i) State the domain and range of $y = 5\cos^{-1} x$. **2**

(ii) Hence sketch $y = 5\cos^{-1} x$. **1**

(k) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x \cos x \, dx$. **2**

(l) Show that $\cos\left\{\tan^{-1}\left[\sin\left(\cot^{-1} x\right)\right]\right\} = \sqrt{\frac{x^2+1}{x^2+2}}$. **2**

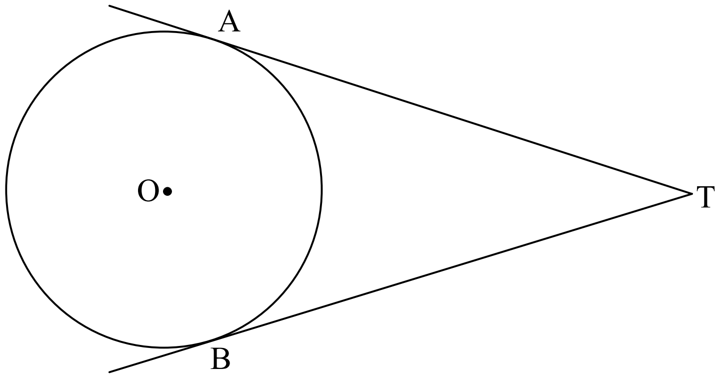
End of Question 1

Start a new examination booklet

QUESTION 2 (23 marks) **Marks**

(a) Evaluate $\lim_{x\to\infty} \tan^{-1}\left(x - x^2\right)$ **1**

(b)



TA and TB are tangents to a circle centre O.

(i) Prove that ATBO is a cyclic quadrilateral. **2**

(ii) Hence or otherwise prove that $\angle \text{ATB} = 2\angle \text{ABO}$. **2**

(c) It has been proposed that for all positive integer values of n the following statement is true.

$$2 + 4 + 6 + ... + 2n = (n+2)(n-1)$$

(i) Assume that the statement is true for $n = k$ and show that it true for $n = k+1$. **2**

(ii) Is the original statement true for all positive integer values of n? Explain your answer. **1**

(d) A spot of light, undergoing simple harmonic motion on a computer screen, has a period of π seconds. When the spot is $3\sqrt{3}$ centimetres to the left of its equilibrium point it has a velocity of 6 centimetres per second towards its equilibrium point.

(i) Show that $x(t) = A\sin(nt) + B\cos(nt)$ is a solution of the simple harmonic motion equation $\ddot{x}(t) = -n^2 x(t)$. **2**

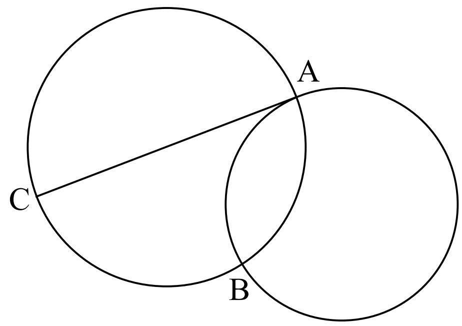
(ii) Using the initial conditions of the spot of light find values for n, A and B to show that the function $x(t) = 3\sin(2t) - 3\sqrt{3}\cos(2t)$ describes the motion of the spot. **3**

(iii) At what time will the spot first reach it's equilibrium point? **2**

(iv) At what time will the spot first reach it's maximum amplitude? **2**

(v) What is the maximum amplitude of the spot of light? **1**

(e)



The diagram shows two circles intersecting at A and B. The diameter of one circle is AC. Copy the diagram into your examination booklet.

(i) On your diagram draw a straight line through A, parallel to CB, to meet the second circle at D. **1**

(ii) Prove that BD is a diameter of the second circle. **2**

(iii) Suppose that BD is parallel to CA. Prove that the circles have equal radii. **2**

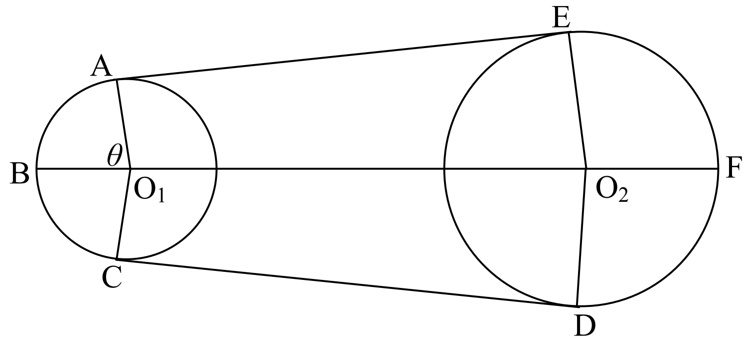
End of Question 2

Start a new examination booklet

QUESTION 3 (22 marks) **Marks**

(a) Evaluate $4\int_0^r \sqrt{r^2 - x^2}\,dx$ using the substitution $x = r\cos\theta$. **3**

(b)



A belt ABCDEF of length 40 m passes round two pulleys, centres O_1 and O_2. The parts ABC and DEF of the belt are in contact with the pulleys and the parts AE and CD are straight.

(i) If $O_1A = 2$ m, $O_2D = 5$ m, and $\angle AOB = \theta$ radians, show that **3**

$$3\tan\theta = 3\theta + 20 - 5\pi .$$

(ii) Using a first approximation of $\frac{\pi}{3}$ and one applications of Newton's method, find a root of the equation to 2 decimal places. **2**

(c) A particle is moving such that its velocity, v m/s, is related to its displacement, x metres, by the formula $v = 2x - 3$.

(i) Show that the acceleration, a m/s^2, of the particle is given by $a = 4x - 3$. **2**

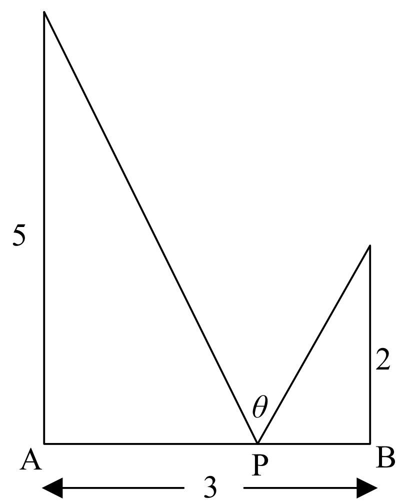
(ii) Initially the particle has displacement $x = 3.5$ m. Show that the relationship between displacement, x, and time, t seconds, can be expressed as

$$\log_e\left(\frac{2x-3}{4}\right) = 2t.$$ **3**

(iii) Write down the function for acceleration in terms of t. **1**

(d) Use mathematical induction to show that $n! \geq 2^{n-1}$ for $n = 1, 2, 3, \ldots$ **4**

(e) How far from A should the point P be, on the interval AB, so as to maximise the angle θ. **4**



End of Question 3

End of Exam

2007 Mathematics Extension 1 Assessment 3: **Solutions Question 1**

1. (a) Evaluate $\cos^{-1}\left(\frac{2}{5}\right)$ in radians to 4 significant figures. [1]

Solution: $1.15927948072740859984658379 4\ldots \approx 1.159$.

(b) Use the table of integrals to find

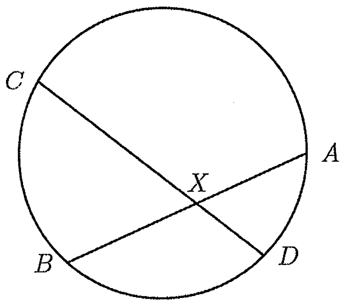
(i) $\int \frac{dx}{9+x^2}$ [1]

Solution: $\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$.

(ii) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{7}\right)$ [1]

Solution: $\frac{d}{dx}\left(\sin^{-1}\frac{x}{7}\right) = \frac{1}{\sqrt{49-x^2}}$.

(c) [1]



If $AX = 3$, $BX = 4$ and $CX = 6$, find DX.

Solution: $DX \times XC = AX \times XB$ (product of intercepts theorem).

$DX \times 6 = 3 \times 4,$

$\therefore\ DX = 2.$

(d) Find the inverse functions of the following

(i) $f(x) = \dfrac{x+1}{4}$ [1]

Solution:
$$\begin{aligned} x &= \frac{f^{-1}(x)+1}{4}, \\ 4x &= f^{-1}(x)+1, \\ f^{-1}(x) &= 4x-1. \end{aligned}$$

(ii) $g(x) = \dfrac{x+2}{1-x}$ [2]

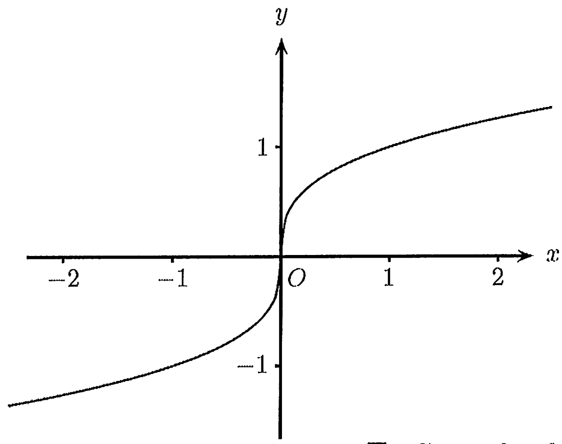
Solution:
$$\begin{aligned} x &= \frac{g^{-1}(x)+2}{1-g^{-1}(x)}, \\ x\left(1-g^{-1}(x)\right) &= g^{-1}(x)+2, \\ x-2 &= g^{-1}(x)+xg^{-1}(x), \\ g^{-1}(x) &= \frac{x-2}{x+1}. \end{aligned}$$

(e) Evaluate $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$. [1]

Solution: $\sin\dfrac{2\pi}{3} = \sin\dfrac{\pi}{3}$ and the range of inverse sine is from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$,
$\therefore\ \sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \dfrac{\pi}{3}$.

(f) It is obvious that $\sqrt{4} < \sqrt{7} < \sqrt{9}$. Use two applications of the 'halving the interval' method on the equation $x^2 - 7 = 0$ to find a smaller interval containing $\sqrt{7}$. [2]

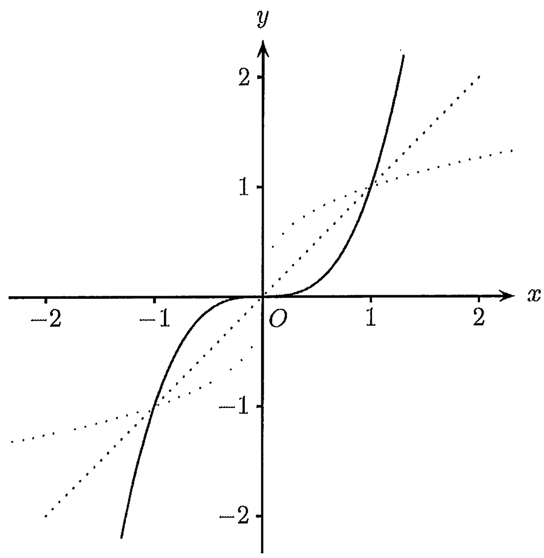
Solution:
$$\begin{aligned} \text{Put } f(x) &= x^2-7, \\ f(2) &= -3, \\ f(3) &= 2, \\ f(2.5) &= -0.75, \\ (3+2.5)/2 &= 2.75, \\ f(2.75) &= 0.5625. \end{aligned}$$
$\therefore$ a better estimate is between 2.5 and 2.75.

(g) 

2

The diagram shows the graph of $y = \sqrt[3]{x}$. Copy the diagram into your writing booklet. On your diagram sketch the graph of the inverse function of $y = \sqrt[3]{x}$.

Solution:



(h) Find $\int 8x(x^2+2)^3\,dx$ using the substitution $u = x^2 + 2$. 2

Solution:

$$\begin{aligned}\int 8x(x^2+2)^3\,dx &= \int 4u^3\,du,\\ &= u^4 + c,\\ &= (x^2+2)^4 + c.\end{aligned}$$

$$\begin{aligned} u &= x^2+2\\ \frac{du}{dx} &= 2x\end{aligned}$$

(i) A rock is thrown into a still pool and causes a circular ripple. The radius of the ripple is increasing at 0.5 m/s.

(i) Find the rate of change of the area as a function of the radius. [2]

Solution: Area, $A = \pi r^2$,
$$\frac{dA}{dr} = 2\pi r.$$
$$\frac{dr}{dt} = 0.5\,\text{m/s (given)}$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt},$$
$$= 2\pi r \times 0.5,$$
$$= \pi r.$$

(ii) How fast is the area increasing when the radius is 10 m? [1]

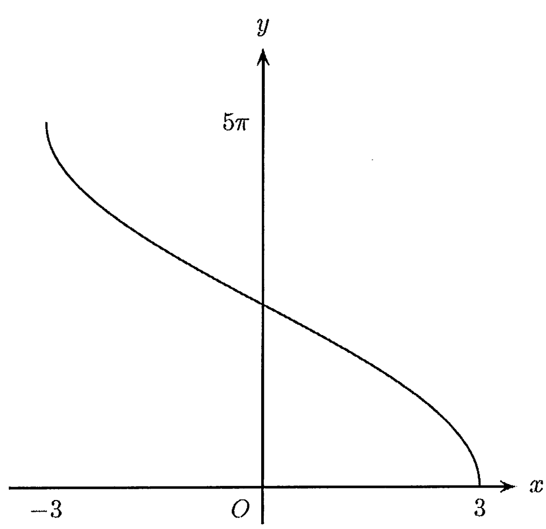
Solution: The area is increasing at $10\pi\,\text{m}^2/\text{s}$, or about $31.4\,\text{m}^2/\text{s}$.

(j) (i) State the domain and range of $y = 5\cos^{-1}\frac{x}{3}$. [2]

Solution:
$$-1 \le \frac{x}{3} \le 1,$$
$$-3 \le x \le 3.$$
$$0 \le y \le 5\pi.$$

(ii) Hence sketch $y = 5\cos^{-1}\frac{x}{3}$. [1]

Solution:



(k) Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x\,\cos x\,dx.$ [2]

Solution: Method 1:

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x\,\cos x\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x\,dx,\\ &= \left[\frac{-\cos 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\\ &= -\tfrac{1}{2}\{-1-0\},\\ &= \tfrac{1}{2}.\end{aligned}$$

Solution: Method 2:

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin x\,\cos x\,dx &= \int_{\frac{1}{\sqrt{2}}}^{1} 2u\frac{du}{dx}.dx,\\ &= [u^2]_{\frac{1}{\sqrt{2}}}^{1},\\ &= 1-\tfrac{1}{2},\\ &= \tfrac{1}{2}.\end{aligned}$$

Put $u = \sin x$,

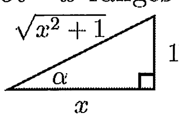
$\dfrac{du}{dx} = \cos x.$

When $x = \frac{\pi}{2}$ $\quad u = 1,$

$x = \frac{\pi}{4}$ $\quad u = \frac{1}{\sqrt{2}}.$

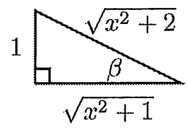
(l) Show that $\cos\left\{\tan^{-1}\left[\sin\left(\cot^{-1} x\right)\right]\right\} = \sqrt{\dfrac{x^2+1}{x^2+2}}.$ [2]

Solution: We note that $x \in \mathbb{R}$ and $\cot^{-1} x$ ranges over the interval $(0,\,\pi)$. Over this domain $\sin\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$.



From the diagram, $\sin\alpha = \dfrac{1}{\sqrt{x^2+1}}$, with range $0 < \sin\alpha < 1$.

Over this domain, putting $\beta = \tan^{-1}(\sin\alpha)$, $0 < \beta < \frac{\pi}{4}$.



Now $\cos\beta = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$, *i.e.* $\cos\left\{\tan^{-1}\left[\sin\left(\cot^{-1} x\right)\right]\right\} = \sqrt{\dfrac{x^2+1}{x^2+2}}.$

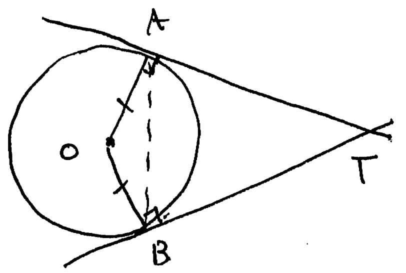
Question (2)

(a) $\lim_{x \to \infty} \tan^{-1}(x - x^2)$

$= \lim_{x \to \infty} \tan^{-1}(-x^2)$

$= -\frac{\pi}{2}$ (1)

(b)



(2)

(i) $\angle OAT = \angle OBT = 90°$

(tgt perpendicular to the radius at the pt of contact)

$\therefore$ OATB is cyclic

(opp. $\angle$'s of OATB are supplementary.

(ii) Join AB, let $\angle OBA = \alpha$

$\therefore \angle OAB = \alpha$ ($\triangle OAB$ is isosceles $OA = OB$)

$\therefore \angle AOB = 180 - 2\alpha$ ($\angle$ sum of $\triangle$)

$\therefore \angle ATB = 2\alpha$ (opp. $\angle$s of a cyclic quad. are supplementary.)

(2) i.e $\angle ATB = 2\angle ABO$.

(c) (i) $2 + 4 + \cdots + 2K + 2(K+1)$

(2) $= (K+2)(K-1) + 2(K+1)$

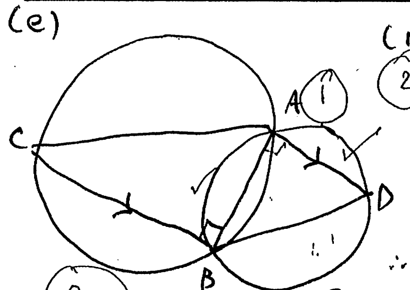
$= K^2 + 3K = [(K+1)+2][(K+1)-1]$

$\therefore$ $S(K+1)$ is true assume $S(K)$ is true.

(1) (ii) not true for $n = 1$, or 2 ..

$n = 1$, L.H.S $= 2$, RHS $= (1+2)(1-1) = 0 \neq$ L.H.S.

(e)



(i) (2)

(ii) $\because$ AC is diameter (2)

$\therefore \angle ABC = 90°$

($\angle$ in a semi circle)

$\therefore \angle DAB = 90°$

(alt. $\angle$'s, AD ∥ BC)

$\therefore$ BD is diameter.

every right angle is an angle in a semi circle.

(iii) CADB is a parm $\therefore CA = BD \Rightarrow$ equal radii

(d)

(i) $x(t) = A\sin(nt) + B\cos(nt)$

$\dot{x} = nA\cos(nt) - nB\sin(nt)$

$\ddot{x} = -n^2 A\sin(nt) - n^2 B\cos(nt)$

$\ddot{x} = -n^2[A\sin(nt) + B\cos(nt)]$

$= -n^2 x(t)$. (2)

(ii) $T = \frac{2\pi}{n}$ $\quad \pi = \frac{2\pi}{n}$

(α) $\therefore n = 2$.

Let $x(t) = R\sin(2t + \alpha)$

expanding :

$= (R\cos\alpha)\sin 2t + (R\sin\alpha)\cos 2t$

A $\quad$ B.

$\therefore A = R\cos\alpha = 3$

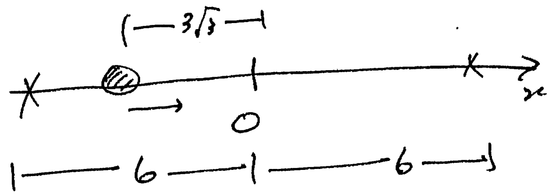
$B = R\sin\alpha = -3\sqrt{3}$. (3)

$\Rightarrow \tan\alpha = -\sqrt{3} \Rightarrow \alpha = -\frac{\pi}{3}$.

$\therefore R = 6$.

$x(t) = 6\sin(2t - \frac{\pi}{3})$

$\dot{x}(t) = 12\cos(2t - \frac{\pi}{3})$.



(β) When $x(t) = 0$ (2)

$0 = \sin(2t - \frac{\pi}{3})$

$\therefore 2t - \frac{\pi}{3} = 0, \pi, 2\pi, \ldots$

$2t = \pi/6 \Rightarrow t = \frac{\pi}{6}$.

(γ) When $x(t) = 6$

$6 = 6\sin(2t - \frac{\pi}{3})$ (2)

$\therefore \sin(2t - \frac{\pi}{3}) = 1$

$\therefore 2t - \frac{\pi}{3} = \frac{\pi}{2}$

$\therefore 2t = \frac{5\pi}{6} \Rightarrow t = \frac{5\pi}{12}$.

(δ) $R = 6$. (1)

QUESTION 3

3(a) $x = r\cos\theta \quad \frac{dx}{d\theta} = -r\sin\theta$

$x=0 \quad \theta = \frac{\pi}{2}$
$x=r \quad \theta = 0$

$4\int_0^r \sqrt{r^2-x^2}\,dx$

$= 4\int_{\frac{\pi}{2}}^{0} \sqrt{r^2-r^2\cos^2\theta}\,.\,-r\sin\theta\,d\theta$

$= 4\int_0^{\frac{\pi}{2}} r\sin\theta\,.\,r\sin\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} 4r^2\sin^2\theta\,d\theta = 4r^2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta$

$= 4r^2\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= 4r^2\left[\frac{\pi}{4} - 0 - (0)\right] = \pi r^2$

(b)(i) $V = 2x-3 \quad \frac{1}{2}v^2 = \frac{1}{2}(4x^2-12x+9)$

$\frac{d(\frac{1}{2}v^2)}{dx} = a = 4x-6$

(ii) $v = \frac{dx}{dt} = 2x-3$

$\frac{dt}{dx} = \frac{1}{2x-3}$

$t = \frac{1}{2}\log_e(2x-3) + C$

$t=0 \quad x = 3.5 \quad C = -\frac{1}{2}\log 4$

$t = \frac{1}{2}\log_e\left(\frac{2x-3}{4}\right)$

$2t = \log_e\left(\frac{2x-3}{4}\right)$

$\frac{2x-3}{4} = e^{2t}$

$2x-3 = 4e^{2t}$

(iii) $a = 4x-6$

$= 2\times 4e^{2t}$

$= 8e^{2t}$

(c) Assume P(k) is true

(i) $k! \geq 2^{k-1}$ for $k = 1,2,3\ldots$

$1! \geq 2^0 \therefore P(1)$ is true

if $n = k+1$:

$(k+1)! = (k+1)k!$

$\geq (k+1)2^{k-1}$ using assumption

$\geq 2.2^{k-1}$ as $k \geq 1$

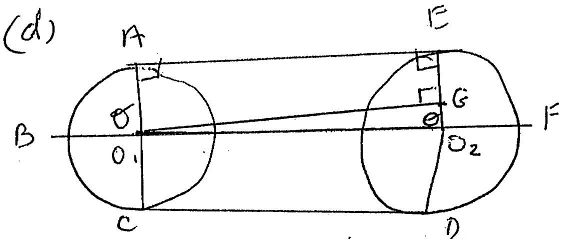
$(k+1)! > 2^k$

$\therefore$ if $P(k)$ is true, $P(k+1)$ is true.

by Mathematical Induction

$n! \geq 2^{n-1}$ for $n = 1,2,3\ldots$

(d)



$BO_2E = \theta$ (cointerior to AO_1O_2)

Draw $O_1G \parallel AE$

$\tan\theta = \frac{O_1G}{O_2G} = \frac{AE}{5-2} = \frac{AE}{3}$

$AE = 3\tan\theta$

$l = r\theta$: Arc $ABC = 2\times 2\theta = 4\theta$

Arc $EFD = 5[2(\pi-\theta)]$

$= 10\pi - 10\theta$

Belt = Arc ABC + 2AE + ARC EFD = 40m

$4\theta + 6\tan\theta + 10\pi - 10\theta = 40$

$6\tan\theta = 6\theta + 40 - 10\pi$

$3\tan\theta = 3\theta + 20 - 5\pi$

(d). (ii)

$$a_1 = a - \frac{f(\theta)}{f'(\theta)}$$

$f(\theta) = 3\tan\theta - 3\theta - 20 + 5\pi$

$f'(\theta) = 3\sec^2\theta - 3$

$$a_1 = \frac{\pi}{3} - \frac{f(\frac{\pi}{3})}{f'(\frac{\pi}{3})}$$

$$= \frac{\pi}{3} - \left(\frac{-2.24}{9}\right)$$

$$= 1.295 \ (3 \ d.p)$$

(e) $\theta = \pi - \tan^{-1}\left(\frac{5}{x}\right) - \tan^{-1}\left(\frac{2}{3-x}\right)$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{5}{x}\right)^2} \times -\frac{5}{x^2} - \frac{2}{1+\left(\frac{2}{3-x}\right)^2} \times -\frac{1}{(3-x)^2} \times -1$$

$$= \frac{5}{x^2+25} - \frac{2}{(3-x)^2+4}$$

$\frac{d\theta}{dx} = 0$ when $5\left[(3-x)^2+4\right] = 2x^2+50$

$65 - 30x^2 + 5x^2 = 2x^2 + 50$

$3x^2 - 30x + 15 = 0$

$x^2 - 10x + 5 = 0$

$$x = \frac{10 \pm \sqrt{80}}{2}$$

$$= 5 \pm 2\sqrt{5}$$

$x = 5 - 2\sqrt{5} \quad (x < 3)$

when $x = 0.5$ $\frac{d\theta}{dx} > 0$

$x = 0.6$ $\frac{d\theta}{dx} < 0$

$\therefore$ max when $x = 5 - 2\sqrt{5}$